ON LIVŠIC'S THEOREM, SUPERRIGIDITY, AND ANOSOV ACTIONS OF SEMISIMPLE LIE GROUPS

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Abstract. We prove a generalization of Livšic's Theorem on the vanishing of the cohomology of certain types of dynamical systems. As a consequence, we strengthen a result due to Zimmer concerning algebraic hulls of Anosov actions of semisimple Lie groups. Combining this with Topological Superrigidity, we find a Hölder geometric structure for multiplicity free Anosov actions.

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1. Introduction

During the last decade, Anosov actions of semisimple Lie groups and their lattices have become a focal point in the study of rigidity properties of such groups. Most importantly, local smooth rigidity has been established for various standard algebraic Anosov actions by a number of authors [?, ?, ?, ?, ?, ?, ?, ?]. The conjecture arose that all such actions are essentially C^∞-conjugate to algebraic actions. The proof of this conjecture for the special class of Cartan actions is the goal of the current paper and its sequel [?].

In this paper, we introduce an additional geometric structure for certain types of volume preserving Anosov actions of a connected higher rank semisimple Lie group G of the noncompact type on a closed manifold M. More precisely, we will find a Hölder Riemannian metric, a Hölder splitting $\bigoplus E_\lambda$ of the tangent bundle and a finite dimensional representation π of G such that the elements in a Cartan subgroup of G expand and contract vectors in E_λ *precisely* according to a weight of π. Note that Zimmer's superrigidity theorem for cocycles yields the same conclusion with respect to a measurable Riemannian metric. The difference in regularity however is crucial to our classification of such actions.

One fundamental tool required to obtain these geometric results is a generalization of celebrated work of Livšic on the cohomology of Anosov systems. Livšic showed in particular that an $\mathbb{R}$-valued Hölder cocycle which is measurably cohomologous to the trivial cocycle is Hölder cohomologous to the trivial cocycle [?]. Livšic also obtained results for cocycles taking values in nonabelian Lie groups under the additional assumption that the cocycle evaluated on generators takes values sufficiently close to the identity. Theorem 2.1 and its corollaries provide bundle theoretic versions of Livšic's theorem as we will now explain. Suppose a group G acts on a principal H-bundle $P \to M$ via bundle automorphisms. Since any bundle has a measurable section, such an action is measurably isomorphic to a skew product action on $M \times H$ via a cocycle $\alpha : G \times M \to H$. Different trivializations of P correspond to distinct yet cohomologous cocycles. Note that α is measurably cohomologous to the trivial cocycle precisely when there is a measurable G-invariant section of P. More generally, there is a complete correspondence in the measurable category between bundle theoretic and cohomological statements. This correspondence breaks down in the continuous or smooth category. Indeed, P may not admit any continuous sections, and thus the G action on P may not give rise to a cocycle. Nevertheless, Livšic's theorem generalizes. In particular, if there is a measurable invariant section of P and H is the semidirect product of compact and abelian groups, then there is also an invariant Hölder section.

Our primary application of this bundle theoretic version of Livšic's theorem is a description of the Hölder algebraic hull of certain G actions on the frame bundle, where G is a semisimple Lie group of higher rank without compact factors. Algebraic hulls are invariants of the action, defined in the measurable, continuous, Hölder and smooth categories for any principal bundle action as above as long as the structure group H is an algebraic group. They are the smallest algebraic subgroup of H for which there is a reduction of P of the relevant regularity which is invariant under the G-action. In the measurable case, a trivial algebraic hull is equivalent to the associated cocycle being cohomologically trivial. In [?], Zimmer was able to show that the measurable algebraic hull of a higher rank semisimple Lie group is reductive with compact center. In Theorem 3.1, we strengthen this description

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to include the Hölder algebraic hull of the frame bundle for Anosov actions of higher rank semisimple Lie groups. The proof uses our version of Livšic's theorem as well as arguments from finite dimensional representation theory. To produce the special Hölder Riemannian metric for multiplicity free Anosov actions of a semisimple group, we combine our description of algebraic hulls with Zimmer's Topological Superrigidity Theorem [**?**, **?**]. The latter theorem allows one to find continuous, Hölder or smooth sections of P which transform under the G action essentially according to a finite dimensional representation π of G provided one can find sections of the same regularity of certain associated bundles to P invariant under a parabolic subgroup of G, e.g. Grassmann bundles. In our case, we can use the stable distribution of an Anosov element of G as the Hölder section of a Grassmann bundle.

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2. Livšic's Theorem and the Hölder Algebraic Hull

The main purpose of this section is to prove our generalization of Livšic's theorem, Theorem 2.1. Let us we begin by mentioning some of the basic notions central to our presentation.

2.1. **Preliminaries.** Let G be a Lie group, possibly discrete. Suppose G acts smoothly (at least $C^{1+\theta}$) and locally freely on a manifold M with a norm $\|\cdot\|$ associated to a Riemannian metric. Call an element $g \in G$ *regular* or *normally hyperbolic* if there exist a continuous decomposition of the tangent bundle

$$TM = \tilde{E}_g^+ + \tilde{E}^0 + \tilde{E}_g^-$$

into g-invariant subbundles, and positive constants $\tilde{C} > 1$, $\tilde{A}, \tilde{B} > 0$ such that for every $m \in M$, for every positive integer n, and for every $v \in \tilde{E}_g^{\pm}(m)$,

$$\frac{1}{\tilde{C}}\|v\|e^{-n\tilde{B}} \leq \|Tg^{\mp n}v\| \leq \tilde{C}\|v\|e^{-n\tilde{A}},$$

and such that E^0 is the tangent distribution of the G-orbits. Call a G action *Anosov* or *normally hyperbolic* if it contains a normally hyperbolic element. While $\tilde{E}_g^-$ is clearly contained in the stable distribution E_g^- of g, E_g^- may also contain elements tangent to the orbits of G. To analyze this contribution, let q be the dimension of G. Consider the q-frames $\mathcal{Q}$ of M tangent to the G-orbits. Fix a basis $X_1, \dots, X_q$ of $\mathfrak{g}$. Since the G-action is locally free, evaluating the X_i defines a smooth section θ of $\mathcal{Q}$. In fact, $\mathcal{Q}$ is a trivial bundle with fiber the full frames of $\mathfrak{g}$, on which G acts via the adjoint representation. Note that θ transforms according to the adjoint transformation for $h \in G$,

$$Th(\theta(m)) = Ad(h)\theta(hm),$$

where Th denotes the derivative of h.

For g a normally hyperbolic element of the G action as above, let $\mathcal{O}_g^-(m)$ correspond to the sum of the generalized eigenspaces of $Ad(g)$ of eigenvalue of modulus less than 1. Then

$$E_g^- = \tilde{E}_g^- \oplus \mathcal{O}_g^-$$

is precisely the stable distribution of g on M. Since $\tilde{E}_g^-$ is continuous and $\mathcal{O}_g^-(m)$ is smooth, E_g^- is continuous. Similarly, the unstable distribution E_g^+ is continuous and splits as a sum

$$E_g^+ = \tilde{E}_g^+ \oplus \mathcal{O}_g^+.$$

The neutral distribution E_g^0 corresponds to the sum of the generalized eigenspaces of $Ad(g)$ with eigenvalue of modulus 1. Since the latter form a Lie subalgebra of $\mathfrak{g}$, it follows that E_g^0 is an integrable distribution. In particular, g is normally hyperbolic to the orbit foliation of a subgroup of G. Finally, there are constants $C > 1$, $A, B > 0$ such that for every $m \in M$, for every positive integer n, and for every $v \in E_g^{\pm}(m)$,

$$\frac{1}{C}\|v\|e^{-nB} \leq \|Tg^{\mp n}v\| \leq C\|v\|e^{-nA}. \tag{1}$$

If M is compact, these notions do not depend on the ambient Riemannian metric. Note that the splitting and the constants in the definition above depend on the normally hyperbolic element. It is well known that the distributions E_g^+ and E_g^- are integrable and are tangent to W_g^s and W_g^u, the stable and unstable foliations of g. In particular, $W_g^-(x)$ denotes the stable manifold through $x \in M$. This is a Hölder foliation whose leaves are smoothly immersed submanifolds of M [**?**, **?**, **?**]. Suppose additionally that an Anosov action preserves a volume.

Then the stable and unstable foliations are absolutely continuous. This follows from [**?**, Theorem 2.1] since the normally hyperbolic element $g \in G$ expands the neutral distribution E^0_g subexponentially by our analysis above. It follows from the usual Hopf argument and the fact that the neutral foliation is contained in the G-orbits that the G-action is ergodic.

Let $P \to M$ be a principal bundle over M with structure group H, a Lie group, and suppose G is a group. We say that G acts on P via *bundle automorphisms* if the G action on P factors to a G action on M. Fix an H-invariant metric on P. Then we say that G acts via *Hölder bundle automorphisms* if the G action on P projects to a smooth (at least $C^{1+\theta}$) action on M and if each element of G is a Hölder homeomorphism of P.

Let us discuss the relationship between bundle automorphisms and cocycles. First, we recall the notion of a cocycle. Suppose that G is a group which acts on a manifold M, and H is a Lie group. A function $\alpha : G \times M \to H$ is called a *cocycle* if it satisfies the cocycle identity:

$$\alpha(g_1 g_2, m) = \alpha(g_1, g_2 m)\alpha(g_2, m) \text{ for all } g_1, g_2 \in G, \text{ and } m \in M.$$

When dealing with measurable cocycles, we require that the cocycle identity hold only for almost all $m \in M$. Two cocycles $\alpha, \beta : G \times M \to H$ are called *equivalent* or *cohomologous* if there exists a function $\phi : M \to H$ such that $\phi(gm)^{-1}\alpha(g,m)\phi(m) = \beta(g,m)$. The regularity of ϕ determines the regularity of the equivalence. In particular, we say that a cocycle is measurably (Hölder, smoothly, etc.) *trivial* if it is measurably (Hölder, smoothly, etc.) equivalent to the trivial cocycle.

Given a cocycle $\alpha : G \times M \to H$, one can construct a G action on the trivial bundle $P = M \times H$ via bundle automorphisms by defining

$$g(m,h) = (mg, \alpha(g,m)h).$$

Note that the regularity of the G action on P will be the same as that of the cocycle α. In particular, if α is Hölder, then the G action on P will be via Hölder bundle automorphisms.

Conversely, if G acts on P via bundle automorphisms, then with respect to any trivialization of P (e.g., a measurable one), one can construct a cocycle which describes this action. In particular, if $\sigma : M \to P$ is a section, then there exists a cocycle $\alpha : G \times M \to H$ so that $g\sigma(m) = \sigma(gm)\alpha(g,m)$. We refer to α as the *cocycle corresponding to* σ. Although different trivializations yield different yet cohomologous cocycles, we shall abuse notation and refer to this class of cocycles as the cocycle for the G action on P.

We now state our main result, and then list a number of consequences.

Theorem 2.1. *Let $P \to M$ be a principal H bundle over a compact connected manifold M where $H = \mathcal{K} \ltimes \mathcal{A}$, the semidirect product of a compact group with an abelian group. Suppose G is a Lie group that acts via Hölder bundle automorphisms on P such that the G action on M is Anosov with $a \in G$ normally hyperbolic. Let V be a transitive left H space which admits an H-invariant metric. Then any G-invariant measurable section of the associated bundle $E_V \to M$ is actually Hölder.*

Remark 2.2. Just as in the case of Liv̌sic's original work, there exist C^1 and C^∞ versions of this result. We will pursue this elsewhere [**?**].

Remark 2.3. A key element in the proof lies in estimating the effects of conjugation by elements in H, cf. Equation 5. We remark that this theorem will hold whenever an appropriate bound on conjugation can be made. As Livšic does in his work, if we assume that the cocycle corresponding to some bounded section takes values sufficiently close to the identity, then it is possible to produce such a bound. This yields the following result.

Corollary 2.4. *Let $P \to M$ be a principal H bundle over a compact connected manifold M where H is any algebraic group. Suppose G is a Lie group that acts via Hölder bundle automorphisms on P such that the G action on M is Anosov with $a \in G$ normally hyperbolic.*

($\star$) *Assume there exists a measurable section $\sigma : M \to P$ taking values in a compact subset $K \subset P$ so that for the corresponding cocycle $\alpha : G \times M \to H$, for some choice of inner product on $\mathfrak{h}$, and for some sufficiently small $\eta > 0$, the operator norm of $Ad(\alpha(a,x))$ is less than $1 + \eta$.*

Let V be a transitive left H space which admits an H-invariant metric. Then any G-invariant measurable section of the associated bundle $E_V \to M$ is actually Hölder.

Call a set $U \subset M$ a *Hölder (measurable) generic* set if U contains an open dense (conull) set. Now suppose G acts by automorphisms of a principal H-bundle $P \to M$ where H is an algebraic group. Assume that the

G action on M is ergodic with respect to some invariant measure μ. An algebraic subgroup $L \subset H$ is called a *Hölder (measurable) algebraic hull* for the G action on P if

(a) there exist a G-invariant Hölder (measurable) generic set $U \subset M$ and a G-invariant Hölder (measurable) section of $E_{H/L}\mid_U \to U$, and
(b) the first assertion is false for any proper algebraic subgroup of L.

Algebraic hulls exist and are unique up to conjugacy. See [**?**, **?**, **?**] for discussions of the properties of algebraic hulls and some of their geometric consequences. In a sense, the algebraic hull describes how much of H is involved in the G action. We remark that we can also define these terms in the C^r and Lipschitz categories in the obvious fashion.

Of course, if H admits a bi-invariant metric, then the homogeneous space H/H_1 for $H_1 \subset H$ admits a left invariant metric. Thus, Theorem 2.1 has an immediate application for algebraic hulls.

Corollary 2.5. *Let $P \to M$ be a principal H bundle over a compact connected manifold M where $H \subset GL(n, \mathbb{R})$ is algebraic. Suppose G acts via bundle automorphism on P such that the G action on M is Anosov. If H admits a bi-invariant metric, then the measurable algebraic hull for the G action on P and the Hölder algebraic hull for the G action on P are equal.*

2.2. **Proof of Theorem 2.1.** The basic strategy of the proof is similar to that of Livšic. A significant difference in our proof is that, unlike in Livšic's situation, we do not have a globally defined section of P giving rise to a cocycle. As it turns out, it is sufficient to pick sections defined on a finite family of open sets which cover M, and study an analog of the cocycle for those sections. The key to obtaining the estimate in Equation 8 lies in estimating the effects of conjugation in H by certain elements.

We begin by introducing some notation. Suppose G acts by automorphisms of a principal H-bundle $P \to M$, and V is an H-space with $E_V \to M$ the associated bundle, i.e., $E_V = (P \times V)/H$. Let $C^r(M; E_V)$ be the set of C^r sections of the associated bundle E_V over M. The following is a well-known result which can be found in [**?**].

Proposition 2.6. *There exists a natural bijective correspondence between H-equivariant C^r maps $P \to V$ and $C^r(M; E_V)$, with respect to which H-equivariant G-invariant C^r maps $P \to V$ correspond to G-invariant elements of $C^r(M; E_V)$. Further, similar results hold if we replace C^r with Hölder, measurable, etc.*

By Proposition 2.6, the assumption that there exists a measurable G-invariant section of the bundle $E_V \to M$ is equivalent to the existence of a measurable G-invariant H-equivariant map $\Phi : P \to V$. Since V is a transitive H space with an H-invariant metric, we may write $V \cong H_v\backslash H$ where H_v is the isotropy of H for some fixed $v \in V$. Note that H_v is compact. We may assume that Φ is defined on a G and H-invariant set. To prove the theorem, we will show that Φ can be extended to all of P in a Hölder manner. The proof consists of two parts. In the first part, we analyze the action of a regular element $a \in G$ on P over a stable manifold of a in M. This leads to the crucial estimate, Equation 8. Then, using this estimate, we adapt Livšic's methods to obtain our result.

Fix a Riemannian metric $\|\cdot\|_M$ on M and an H-invariant Riemannian metric $\|\cdot\|_P$ on P. Choose finitely many open neighborhoods $U_j \subset M$ and a compact set $K \subset P$ such that

(a) $\cup_j U_j = M$,
(b) there exists $\zeta > 0$ such that the ζ ball around any point in M is contained in some U_j, and
(c) there exist smooth sections $s_j : U_j \to K \subset P$ with uniform Lipschitz constant δ.

For any $y^* \in P$ lying in the fiber over $y \in U_i \subset M$ there exists a unique $h_i(y^*)$ such that $y^* = s_i(y)h_i(y^*)$. Note that h_i varies as smoothly as y^* does, for all y^* lying over U_i, and that the h_i's are uniformly Lipschitz on any set of the form $s_i(U_i) \cdot K_0$ where $K_0 \subset H$ is a compact subset.

Using the definition of a normally hyperbolic diffeomorphism in Equation 1, there exists some k such that $Ce^{-kA} < 1$. For ease of notation, we will replace a^k with a and kA with A, i.e., we will assume that $Ce^{-A} < 1$.

Let $x \in M$. Then by our choice of the U_j's, for every m there exists some j such that the ζ ball about $a^m x$ lies entirely within U_j. For any x and m, pick $i(m, x)$ to be such a j. We remark that although there is no canonical choice for these $i(m, x)$, the calculations we desire will not depend on these choices. These $h_{i(m,x)}(a^m x^*)$ will serve as our cocycle analogs, and we want to estimate $h_{i(m,x)}(a^m x^*)$ as x varies along a stable manifold of a by expanding $h_{i(m,x)}(a^m x^*)$ as a product where each factor will be controllable. To this end, we set $q(a, 0, x^*) = h_{i(0,x)}(x^*)$ and $q(a, j, x^*) = h_{i(j,x)}(a^j x^*)[h_{i(j-1,x)}(a^{j-1}x^*)]^{-1}$ for $j \geq 1$. Hence, we can write

$$h_{i(j,x)}(a^j x^*) = q(a, j, x^*)q(a, j-1, x^*) \cdots q(a, 0, x^*) \qquad (2)$$

for any $j > 0$.

With the aim of deriving some preliminary properties of the $q(a, j, x^*)$, we use the definition of the h_i's to obtain

$$\begin{aligned} a^j x^* &= a[a^{j-1}x^*] = a[s_{i(j-1,x)}(a^{j-1}x)h_{i(j-1,x)}(a^{j-1}x^*)] \\ &= [as_{i(j-1,x)}(a^{j-1}x)]h_{i(j-1,x)}(a^{j-1}x^*) \\ &= [s_{i(j,x)}(a^j x)h_{i(j,x)}(as_{i(j-1,x)}(a^{j-1}x))]h_{i(j-1,x)}(a^{j-1}x^*). \end{aligned}$$

By uniqueness, we conclude

$$h_{i(j,x)}(a^j x^*) = h_{i(j,x)}(as_{i(j-1,x)}(a^{j-1}x)) \cdot h_{i(j-1,x)}(a^{j-1}x^*),$$

and therefore, for any $j > 0$,

$$\begin{aligned} q(a, j, x^*) &= h_{i(j,x)}(a^j x^*)[h_{i(j-1,x)}(a^{j-1}x^*)]^{-1} \\ &= h_{i(j,x)}(as_{i(j-1,x)}(a^{j-1}x))h_{i(j-1,x)}(a^{j-1}x^*)[h_{i(j-1,x)}(a^{j-1}x^*)]^{-1} \\ &= h_{i(j,x)}(as_{i(j-1,x)}(a^{j-1}x)). \end{aligned}$$

So, although $q(a, j, x^*)$ is defined as a function of x^*, for $j > 0$ it actually depends only on x and our choice of $i(m, x)$. Of course, $q(a, 0, x^*) = h_{i(0,x)}(x^*)$ still depends on x^*. For $j > 0$, let us define $q(a, j, x) = q(a, j, x^*)$. So, for $j > 0$, we get

$$q(a, j, x) = h_{i(j,x)}(as_{i(j-1,x)}(a^{j-1}x)). \tag{3}$$

Since the s_i's take values in a compact set, it follows that there exists a compact set $K_1 \subset H$ such that $K_1^{-1} \subset K_1$ and $q(a, j, x) \in K_1$ for every $j > 0$ and for every $x \in M$. Without loss of generality we may assume that $\mathcal{K} \times \{0\} \subset K_1$.

Choose $y \in W_a^s(x)$ and let $y^* \in P$ lie in the fiber over y. Using Equation 1, we get

$$d_M(a^j x, a^j y) < Ce^{-jA}d_s(x, y),$$

where d_M is the metric in M, and d_s is the induced metric on the leaves of the stable foliation. In particular, if $d_s(x, y) < \zeta$, then by choice of $i(j, x)$, $a^j y \in U_{i(j,x)}$ for all $j \geq 0$. Pick an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}$, the Lie algebra of H, which is invariant under the compact group $Ad(H_v)$, and let d_H be the corresponding left invariant Riemannian metric on H.

Since the h_i's are uniformly Lipschitz on any set of the form $s_i(U_i) \cdot K_0$ where $K_0 \subset H$ is a compact set, and since $aK_0 \subset H$ can be written as a finite union of sets of this form, it follows that the h_i's are uniformly Lipschitz on aK with constant, say λ, which depends only on a. Let c and θ be the Hölder constant and exponent for multiplying K by a in P, so that for any $p_1, p_2 \in K$, we have $d_P(ap_1, ap_2) \leq cd_P(p_1, p_2)^\theta$. If we let $\Delta = \lambda c\delta^\theta C^\theta$, then for all $j > 0$,

$$\begin{gathered} d_H(q(a, j, x^*), q(a, j, y^*)) \\ \leq d_H(h_{i(j,x)}(as_{i(j-1,x)}(a^{j-1}x)), h_{i(j,x)}(as_{i(j-1,x)}(a^{j-1}y))) \\ \leq \lambda d_P(as_{i(j-1,x)}(a^{j-1}x), as_{i(j-1,x)}(a^{j-1}y)) \\ \leq \lambda cd_P(s_{i(j-1,x)}(a^{j-1}x), s_{i(j-1,x)}(a^{j-1}y))^\theta \\ \leq \lambda c\delta^\theta d_M(a^{j-1}x, a^{j-1}y)^\theta \\ \leq \lambda c\delta^\theta C^\theta e^{-\theta(j-1)A}d_s(x, y)^\theta. \\ = \Delta e^{-\theta(j-1)A}d_s(x, y)^\theta. \end{gathered} \tag{4}$$

It is crucial to note that although Δ depends on a, it does not depend on j.

At this point, we will begin to analyze how $h_{i(n,x)}(a^m x^*)$ varies over a stable manifold. Using Equation 2, the triangle inequality, and left invariance we have

$$\begin{gathered} d_H(h_{i(m,x)}(a^m x^*), h_{i(m,y)}(a^m y^*)) \\ = d_H(q(a, m, x^*) \cdots q(a, 0, x^*), q(a, m, y^*) \cdots q(a, 0, y^*)) \end{gathered}$$

$$\leq \sum_{j=0}^{m} d_H(q(a,m,y^*)\cdots q(a,j+1,y^*)q(a,j,x^*)\cdots q(a,0,x^*),$$
$$q(a,m,y^*)\cdots q(a,j,y^*)q(a,j-1,x^*)\cdots q(a,0,x^*))$$

$$= \sum_{j=0}^{m} d_H(q(a,j,x^*)\cdots q(a,0,x^*), q(a,j,y^*)q(a,j-1,x^*)\cdots q(a,0,x^*))$$

$$= \sum_{j=0}^{m} d_H(C_0\circ\cdots\circ C_{j-1}(q(a,j,x^*)), C_0\circ\cdots\circ C_{j-1}(q(a,j,y^*)))$$

where C_i is conjugation in H by $q(a,i,x^*)$ for $i \geq 0$ and C_{-1} is the identity transformation.

So, in order to obtain an estimate for $d_H(h_{i(m,x)}(a^m x^*), h_{i(m,y)}(a^m y^*))$, we will produce one for $d_H(C_0\circ\cdots\circ C_{j-1}(q(a,j,x^*)), C_0\circ\cdots\circ C_{j-1}(q(a,j,y^*)))$. Recall that by assumption $H = \mathcal{K} \ltimes \mathcal{A}$, where $\mathcal{K}$ is compact and $\mathcal{A}$ is abelian. Without loss of generality we may assume that with respect to the left invariant metric on H, that $\mathcal{K}$ acts via isometries on $\mathcal{A}$. For $k_i \in \mathcal{K}$ and $a_i \in \mathcal{A}$, we have that multiplication is given by $(k_1,a_1)(k_2,a_2) = (k_1k_2, k_2^{-1}a_1 + a_2)$.

Let $q(a,i,x^*) = (r_i, s_i) \in \mathcal{K} \ltimes \mathcal{A}$. Then for $h = (k,a) \in \mathcal{K}\ltimes\mathcal{A}$, we have

$$C_0\circ\cdots\circ C_{j-1}((k,a)) = (r_0,s_0)\cdots(r_{j-1},s_{j-1})(k,a)(r_{j-1},s_{j-1})^{-1}\cdots(r_0,s_0)^{-1}$$

$$= \Big((r_0\cdots r_{j-1})k(r_0\cdots r_{j-1})^{-1}, r_0\Big\{\left[(r_1\cdots r_{j-1})(k^{-1}-Id)(r_1\cdots r_{j-1})^{-1}\right]s_0$$

$$+ r_1\Big\{\left[(r_2\cdots r_{j-1})(k^{-1}-Id)(r_2\cdots r_{j-1})^{-1})\right]s_1 + \cdots + r_{j-1}\Big\{[k^{-1}-Id]s_{j-1} + a\Big\}\cdots\Big\}\Big\}\Big)$$

Since $\mathcal{K}$ is compact, there exists some constant $J > 1$ such that for any $k_1, k_2 \in \mathcal{K}$, $d_H((k_1k_2k_1^{-1},0),(1,0)) \leq J d_H((k_2,0),(1,0))$. Additionally, since K_1 is compact, there exists some constant $L > 1$ such that $d_H((k,a),(1,0)) < L$ for every $(k,a) \in K_1$. In particular, $d_H(q(a,j,x^*),(1,0)) < L$ for all j and almost every $x \in M$. Further, there exists some $\Upsilon > 1$ so that $d_H((1,(k^{-1}-Id)a),(1,0)) \leq \Upsilon d_H((k,0),(1,0)) d_H((1,a),(1,0))$ for all $k \in \mathcal{K}$. Also, there exists some constant $\omega > 1$ such that $d_H((k,0),(1,0)), d_H((1,a),(1,0)) < \omega d_H((k,a),(1,0))$ for all $(k,a) \in K_1$. Finally, by left invariance and the triangle inequality note that $d_H((k,a),(1,0)) \leq d_H((k,0),(1,0)) + d_H(((1,a),(1,0))$.

Using the fact that $\mathcal{K}$ acts via isometries on $\mathcal{A}$, we have for all $j > 0$,

$$\begin{aligned}
&d_H(C_0 \circ \cdots \circ C_{j-1}((k,a)),(1,0)) \\
&\leq d_H(([r_0 \cdots r_{j-1}]k[r_0 \cdots r_{j-1}]^{-1},0),(1,0)) \\
&+ \left(\textstyle\sum_{i=0}^{j-1} d_H(1, r_0 \cdots r_i([r_{i+1} \cdots r_{j-1}(k^{-1} - Id)[r_{i+1} \cdots r_{j-1}]^{-1})s_i,(1,0))\right) \\
&+d_H((1, r_0 \cdots r_{j-1}(a)),(1,0)) \\
&\leq J d_H((k,0),(1,0)) + \left(\textstyle\sum_{i=0}^{j-1} d_H((1,[r_{i+1} \cdots r_{j-1}](k^{-1} - Id)[r_{i+1} \cdots r_{j-1}]^{-1}s_i),(1,0))\right) \\
&+d_H((1,a),(1,0)) \\
&\leq J d_H((k,0),(1,0)) \\
&+ \left(\textstyle\sum_{i=0}^{j-1} \Upsilon d_H((1,[r_{i+1} \cdots r_{j-1}]k[r_{i+1} \cdots r_{j-1}]^{-1}s_i),(1,0)) d_H((1,s_i),(1,0))\right) \\
&+d_H((1,a),(1,0)) \\
&\leq J d_H((k,0),(1,0)) + \left(\textstyle\sum_{i=0}^{j-1} \Upsilon J d_H((k,0),(1,0)) d_H((1,s_i),(1,0))\right) + d_H((1,a),(1,0)) \\
&\leq J d_H((k,0),(1,0)) + \left(\textstyle\sum_{i=0}^{j-1} \Upsilon J L d_H((k,0),(1,0))\right) + d_H((1,a),(1,0)) \\
&\leq (j+2)\Upsilon \omega J L d_H((k,a),(1,0)).
\end{aligned}$$

By Equation 4,

$$d_H(q(a,j,x^*),q(a,j,y^*)) = d_H(q(a,j,y^*)^{-1}q(a,j,x^*),(1,0)) \leq \Delta e^{-\theta(j-1)A} d_s(x,y)^\theta.$$

Hence, we have for almost every $x \in M$ and for $y \in W_a^s(x)$,

$$\begin{aligned}
&d_H(h_{i(m,x)}(a^m x^*), h_{i(m,y)}(a^m y^*)) \\
&\leq \textstyle\sum_{j=0}^{m} d_H(C_0 \circ \cdots \circ C_{j-1}(q(a,j,x^*)), C_0 \circ \cdots \circ C_{j-1}(q(a,j,y^*))) \\
&\leq \textstyle\sum_{j=0}^{m} d_H(C_0 \circ \cdots \circ C_{j-1}(q(a,j,y^*)^{-1}q(a,j,x^*)),(1,0)) \\
&\leq d_H(h_{i(0,x)}(x^*), h_{i(0,y)}(y^*)) + \left(\textstyle\sum_{j=1}^{m}(j+2)\Upsilon\omega J L \Delta e^{-\theta(j-1)A}\right) d_s(x,y)^\theta.
\end{aligned} \tag{5}$$

Since $\theta, A > 0$, $\sum_{j=1}^{\infty}(j+2)\Upsilon\omega J L \Delta e^{-\theta(j-1)A}$ converges absolutely.

Summarizing, there exists a constant $Q > 0$ such that for every $m > 0$, and every $y \in W_\zeta^s(x)$, the ζ-ball in $W^s(x)$ about x,

$$\begin{aligned}
&d_H(h_{i(m,x)}(a^m x^*), h_{i(m,x)}(a^m y^*)) \\
&< d_H(h_{i(0,x)}(x^*), h_{i(0,x)}(y^*)) + Q d_s(x,y)^\theta.
\end{aligned} \tag{6}$$

Similarly, there exists a constant $Q' > 0$ such that for every $m < 0$, and every $y \in W_\zeta^u(x)$, the ζ-ball in $W^u(x)$ about x,

$$\begin{aligned}
&d_H(h_{i(m,x)}(a^m x^*), h_{i(m,x)}(a^m y^*)) \\
&< d_H(h_{i(0,x)}(x^*), h_{i(0,x)}(y^*)) + Q' d_u(x,y)^\theta.
\end{aligned} \tag{7}$$

Without loss of generality we can assume that $Q = Q'$. We may therefore conclude that if x^*, y^* are contained in $K \subset P$ (recall that K is defined on page 4), then there exists a bound κ (depending on K) such that

$$d_H(h_{i(m,x)}(a^m x^*), h_{i(m,x)}(a^m y^*)) < \kappa d_P(x^*, y^*)^\theta \tag{8}$$

provided that $y \in W^s_\zeta(x)$ or $y \in W^u_\zeta(x)$ and $x \in M_s \cap M_u$.

We return to our measurable G-invariant H-equivariant map $\Phi : P \to V$. By G-invariance and H-equivariance, we have

$$\begin{aligned} \Phi(x^*) &= \Phi(a^m x^*) = \Phi(s_{i(m,x)}(a^m x) h_{i(m,x)}(a^j x)) \\ &= [h_{i(m,x)}(a^j x)]^{-1} \Phi(s_{i(m,x)}(a^m x)). \end{aligned} \tag{9}$$

As mentioned above, we need to show that Φ can be extended to a Hölder function on all of P. We will do this by adapting Livšic's argument in [**?**, Theorem 9].

Using Lusin's Theorem [**?**, Theorem 2.24], there exists a compact set $M' \subset M$ with measure $> 1/2$ such that $M' \cap U_i$ is compact and $\Phi \circ s_i : M' \cap U_i \to H$ is uniformly continuous for every i. By H-equivariance, it will suffice to show that Φ is Hölder on K, since K is a compact set in P containing an open set covering all of M.

Let M_0 be the set in M consisting of all x such that

(a) $\displaystyle\lim_{h\to\infty} \frac{1}{h} \sum_{i=0}^{h} \chi_{M'}(a^i x) = \mu(M')$

(b) $\displaystyle\lim_{h\to-\infty} \frac{1}{-h} \sum_{i=h}^{0} \chi_{M'}(a^i x) = \mu(M')$, and

(c) x lies in the projection from P to M of the set on which Φ is G-invariant and H-equivariant.

Let μ_s and μ_u denote the conditional measures on the stable and unstable foliations. Recursively define

$$M_{n+1} = \{x \in M_n | \mu_s(W^s_\zeta(x) \setminus M_n) = \mu_u(W^u_\zeta(x) \setminus M_n) = 0\},$$

and let $M_\infty = \cap_n M_n$. By absolute continuity of the stable and unstable foliations we have $\mu(M_\infty) = 1$. By [**?**], absolute continuity also implies that there exist constants $\xi_1, \omega > 0$ such that if $d_M(x_0, x_5) < \omega$ for $x_0, x_5 \in M_\infty$, then there exist points $x_1, x_2, x_3, x_4 \in M_\infty$ such that

(a) $x_1 \in W^s(x_0), d_s(x_0, x_1) < \xi_1 d_M(x_0, x_5)$,
(b) $x_2 \in W^u(x_1), d_u(x_1, x_2) < \xi_1 d_M(x_0, x_5)$,
(c) $x_3 \in W^s(x_2), d_s(x_2, x_3) < \xi_1 d_M(x_0, x_5)$,
(d) $x_3 \in W^u(x_4), d_u(x_3, x_4) < \xi_1 d_M(x_0, x_5)$, and
(e) x_4 lies in the G orbit of x_5, $d_M(x_4, x_5) < \xi_1 d_M(x_0, x_5)$.

To demonstrate that Φ can be extended to a Hölder function, we want to pick x_0^* and x_5^* lying in the fibers over x_0 and x_5 and show that $d_V(\Phi(x_0^*), \Phi(x_5^*))$ can be bounded by $d_H(x_0^*, x_5^*)^\theta$. This will require the use of Equation 8, which is valid only for points lying in the same stable or unstable manifolds. In addition to using canonical coordinates, we will require our points to lie in M_∞, which necessitates the use of the points x_1, x_2, x_3, and x_4.

For given $x_0^*, x_5^* \in K$ lying in the fibers over x_0, x_5, it is possible to pick $x_i^* \in K, i = 1, 2, 3$ in the fibers over x_i such that

$$d_P(x_i^*, x_{i+1}^*) < \xi_2 d_P(x_0^*, x_5^*)$$

for some $\xi_2 > 1$. Additionally, we may assume without loss of generality that x_4^* has been chosen so that it lies in the G-orbit of x_5^*.

Since Φ is G-invariant, we have $\Phi(x_4^*) = \Phi(x_5^*)$, so that $d_V(\Phi(x_4^*), \Phi(x_5^*)) = 0$. Pick $d_P(x_0^*, x_5^*) < \frac{\zeta}{\xi_2}$. Then

$$d_V(\Phi(x_0^*), \Phi(x_5^*)) \le \sum_{i=0}^{3} d_V(\Phi(x_i^*), \Phi(x_{i+1}^*)). \tag{10}$$

Now, using Equation 9, the triangle inequality, and left invariance, for $i = 0, 1, 2, 3$, we have

$$\begin{aligned} & d_V(\Phi(x_i^*), \Phi(x_{i+1}^*)) \\ &= d_V([h_{i(m,x_i)}(a^m x_i)]^{-1} \Phi(s_{i(m,x_i)}(a^m x_i)), [h_{i(m,x_i)}(a^m x_{i+1})]^{-1} \Phi(s_{i(m,x_i)}(a^m x_{i+1})) \\ &\le d_V(\Phi(s_{i(m,x_i)}(a^m x_i)), \Phi(s_{i(m,x_i)}(a^m x_{i+1}))) \\ &+ d_V(\Phi(s_{i(m,x_i)}(a^m x_{i+1})), [h_{i(m,x_i)}(a^m x_i)][h_{i(m,x_i)}(a^m x_{i+1})]^{-1} \Phi(s_{i(m,x_i)}(a^m x_{i+1}))). \end{aligned}$$

First, note that if i is odd (even) then as $m \to \infty(-\infty)$, $a^m x_i$ and $a^m x_{i+1}$ converge. Thus, using uniform continuity of $\Phi \circ s_i$ on M', it follows that

$$d_V(\Phi(s_{i(m,x_i)}(a^m x_i)), \Phi(s_{i(m,x_i)}(a^m x_{i+1})))$$

can be made as small as we like by choosing m sufficiently large.

Next, using Equation 8, we can bound $d_H(1, [h_{i(m,x_i)}(a^m x_i)][h_{i(m,x_i)}(a^m x_{i+1})]^{-1})$ in terms of $d_P(x_0^*, x_5^*)^\theta$ independently of $m > 0$ ($m < 0$ if i is even). Since $\Phi \circ s_i$ takes values in a compact set, it follows that we can bound

$$d_V(\Phi(s_{i(m,x_i)}(a^m x_{i+1})), [h_{i(m,x_i)}(a^m x_i)][h_{i(m,x_i)}(a^m x_{i+1})]^{-1}\Phi(s_{i(m,x_i)}(a^m x_{i+1})))$$

in terms of $d_P(x_0^*, x_5^*)^\theta$ independently of $m > 0$ ($m < 0$ if i is even).

Hence, for each i, we have a bound for $d_H(\Phi(x_i^*), \Phi(x_{i+1}^*))$ in terms of $d_P(x_0^*, x_5^*)^\theta$. Consequently, there exists a bound for $d_V(\Phi(x_0^*), \Phi(x_5^*))$ in terms of $d_P(x_0^*, x_5^*)^\theta$. That Φ can be Hölder extended to all of K follows using this bound, which completes the proof. □

3. Anosov Actions of Semisimple Groups

Our goal in this section is to combine Theorem 2.1 with Topological Superrigidity to obtain geometric information for Anosov actions of semisimple Lie groups, and, in particular, Theorem 3.6. We begin this section by stating these results as well as briefly describing the relevant definitions and facts necessary to understand them. See [**?**] and [**?**] for a complete discussion of the related concepts and properties. We then continue in the following subsections by proving our results.

3.1. **Statement of Results.** Suppose G is a connected semisimple Lie group without compact factors such that each simple factor of G has $\mathbb{R}$-rank at least 2. Suppose that G acts on a closed manifold M such that the G action is Anosov and volume preserving. Then G acts via derivatives on the general frame bundle over M, and since the action preserves a volume, there exists a G-invariant reduction to a principal H bundle with $H \subset SL(n, \mathbb{R})$, an $\mathbb{R}$-algebraic subgroup. It follows therefore that any algebraic hull of the G action on the general frame bundle is contained in $SL(n, \mathbb{R})$. In fact, we have the following very precise description of the algebraic hull.

Theorem 3.1. *Suppose G is a connected semisimple Lie group without compact factors such that each simple factor of G has $\mathbb{R}$-rank at least 2. Suppose that G acts on a closed manifold M such that the G action is Anosov and volume preserving. Then the Hölder algebraic hull of the G action by derivatives on the frame bundle is reductive with compact center.*

Remark 3.2. This theorem also holds for volume preserving Anosov actions of a cocompact lattice $\Gamma \subset G$ where each simple factor of G has rank 2 or 3. Additionally, this result holds whenever there are enough normally hyperbolic elements in Γ. For instance, if there exists a split Cartan $A \subset G$ such that $A \cap \Gamma$ is cocompact in A and such that $\{\log \gamma \| \gamma \in A \cap \Gamma\}$ is dense in the space of directions of $\mathfrak{a}$, the Lie algebra for A. Or, if Γ is invariant under the Weyl group for A. The main issue is to ensure that an appropriate version of Proposition 3.18 below holds.

Let G act on a set S. Then $s \in S$ is called a *parabolic invariant* if there exists a parabolic subgroup $Q \subset G$ such that Q fixes s. In particular, if G acts by automorphisms of a principal H-bundle $P \to M$, and V is an H-space with $E_V \to M$ the associated bundle, a *parabolic invariant section* is a section of $E_V \to M$ invariant under a parabolic subgroup of G.

Frequently, it is possible to obtain a great deal of geometric information on a generic subset of M. Of course, we also wish to emphasize the distinction between proper generic subsets and all of M. Let G act via principal bundle automorphisms on $P(M, H)$ with Hölder (measurable) algebraic hull L. The G action on $P(M, H)$ is *Hölder (measurably) complete* if there exists a Hölder (measurable) G-invariant section of $E_{H/L}$. Given a principal H-bundle $P \to M$, and V an H-space, a section ϕ of E_V is called *effective for P* if H acts effectively on $\Phi(P)$ where $\Phi : P \to V$ is the H-map corresponding to ϕ. Suppose G acts ergodically and Hölder (measurably) completely on P where H is an algebraic group and V is an algebraic variety. Then a Hölder (measurable) section $\phi : M \to E_V$ is *Hölder (measurably) G-effective* if it is effective for $P_1 \subset P$ where P_1 is a G-invariant reduction to $L \subset H$, the Hölder (measurable) algebraic hull.

Any $\mathbb{R}$-split Cartan $A \subset G$ contains a normally hyperbolic element $a \in A$ (cf. Lemma 3.9). Suppose $\dim(E_a^-) = k$ and let $Gr(k, n)$ be the set of k-dimensional subspaces of $\mathbb{R}^n$. Form the associated bundle $E_{Gr(k,n)} = (P \times$

$Gr(k,n))/SL(n,\mathbb{R})$. Then we can define an A-invariant Hölder section $\phi : M \to E_{Gr(k,n)}$ by setting $\phi(m) = E_a^-(m)$. The following demonstrates that this section actually possesses greater geometric properties.

Proposition 3.3. *Suppose G is a connected semisimple Lie group without compact factors such that each simple factor of G has $\mathbb{R}$-rank at least 2. Suppose G acts on a closed manifold M such that the G action is Anosov and volume preserving. Let H be the Hölder algebraic hull. Then, modulo a compact subgroup of H, there exists a Hölder G-effective parabolic invariant section $\phi : M \to E_{Gr(k,n)}$, i.e., there exist*

1. *a normal subgroup $N \subset H$ which fixes $\Phi(P/N)$,*
2. *a parabolic subgroup $Q \subset G$, and*
3. *a Q-invariant $N\backslash H$-equivariant map $\Phi : P/N \to Gr(k,n)$,*

such that H/N acts effectively on $\Psi(P/N)$.

Let $P \to M$ be a principal H-bundle on which G acts via bundle automorphisms. If $\pi : G \to H$ is a homomorphism, then a section $s : M \to P$ is called *totally π-simple* if for $g \in G$ and $m \in M$,

$$s(gm) = g.s(m).\pi(g)^{-1}.$$

Here, of course, M and P are left G-spaces, and P is a right H-space.

Theorem 3.4 (Topological Superrigidity)**.** *Let G be a connected semisimple Lie group, $\mathbb{R}$-rank $(G) \geq 2$, with G acting via bundle automorphisms on $P(M,H)$, H an algebraic $\mathbb{R}$-group, V an $\mathbb{R}$-variety on which H acts algebraically, and G acting ergodically on M with respect to a probability measure μ where $supp(\mu) = M$. Assume the action is Hölder complete. If there exists a G effective Hölder parabolic invariant section ψ of $E_V \to M$, then, by possibly passing to a finite cover of G, there exist*

1. *a homomorphism $\pi : G \to H$,*
2. *$v_0 \in V$, and*
3. *a totally π-simple Hölder section s of $P \to M$*

such that ψ is the associated section (s, v_0) of E_V, i.e. $\psi(x) = [s(x), v_0]$.

Remark 3.5.
1. Topological Superrigidity was proved by Zimmer in [**?**], and the proof of a generalization appears in [**?**]. The version above differs from the original version of Topological Superrigidity in that we have replaced Hölder functions for C^r functions. However, the proof for Theorem 3.4 requires only a minor modification of the proofs presented in [**?**, **?**].
2. Effectiveness of ψ ensures that we can see enough of H in the image ψ. Note that by passing to a suitable subquotient, it is always possible to obtain an effective section, and, in fact, this procedure is required for most of our applications. More explicitly, let $\Psi : P \to V$ be the H-equivariant map corresponding to ψ, and let $N \subset H$ be the kernel of the H action on $\Psi(P)$. We can then obtain a G-effective section of the V associated bundle to the principal bundle P/N, and by applying the last theorem to this section, we obtain a homomorphism $\sigma : G \to H/N$ and a totally σ-simple Hölder section s of $P/N \to M$. Since H is reductive and N is normal, H is an almost direct product $H = N \cdot H_1$ of N with a normal subgroup $H_1 \subset H$. Then, we can produce a homomorphism $\pi : G \to H_1 \subset H$ which modulo N factors to σ. Then the section s will also be totally π-simple. See [**?**] for further details.

Our main result, which follows, is obtained by combining Proposition 3.3 and Theorem 3.4.

Theorem 3.6. *Suppose G is a connected semisimple Lie group of higher rank without compact factors such that each simple factor of G has $\mathbb{R}$-rank at least 2. Suppose that G acts on a closed manifold M such that the G action is Anosov and volume preserving. Let H be the Hölder algebraic hull of the G action on $P \to M$, the G-invariant reduction of the derivative action on the full frame bundle over M. Then, by possibly passing to a finite cover of G, there exist*

1. *a normal subgroup $K \subset H$,*
2. *a Hölder section $s : M \to P/K$, and*
3. *a homomorphism $\pi : G \to H$ (obtained from a homomorphism $G \to H/K$ as in Remark 3.5.2),*

such that $s(gm) = g.s(m).\pi(g)^{-1}$ for every $g \in G$ and every $m \in M$.

Moreover, if π is multiplicity free, i.e., all irreducible subrepresentations of π have multiplicity one, then $K \subset H$ is a compact normal subgroup.

Corollary 3.7. *Let G, P, M, and H be as in Theorem 3.6. Assume the irreducible subrepresentations of π are multiplicity free so that K is compact. Let A be a maximal $\mathbb{R}$-split Cartan of G, with $\{\chi\}$ the set of weights of π with respect to A. There exist*

1. *a K-invariant Hölder Riemannian metric, $\|\cdot\|_K$, on M, and*
2. *a K-invariant Hölder decomposition $TM = \bigoplus E_\chi$*

such that for every $v \in E_\chi$ and $a \in A$,

$$\|av\|_K = e^{\chi(\log a)}\|v\|_K.$$

Proof. Every frame f over a point $p \in M$ determines an inner product $\langle\cdot,\cdot\rangle_f$ on T_pM by declaring f to be orthonormal. Then a K-orbit of frames fK determines an inner product $\langle\cdot,\cdot\rangle_{fK}$ by averaging the $\langle\cdot,\cdot\rangle_{fk}$ over K. This defines a smooth map from P/K to the bundle of K-invariant inner products on tangent spaces over M. Composing this map with the Hölder section $s : M \to P/K$ determines a K-invariant Hölder Riemannian metric $\|\cdot\|_K$ on M. Pick a measurable section $\sigma : M \to P$ which projects onto s. Then σ is a measurable totally π-simple framing of M. Let $TM = \oplus E_i$ be the Lyapunov decomposition corresponding to the action of A. The exponents for the Lyapunov decomposition correspond to the weights of A under π (cf. e.g. [**?**, **?**, **?**]). Thus we can write $TM = \bigoplus E_\chi$ such that for every $v \in E_\chi$ and $a \in A$, $\|av\|_K = e^{\chi(\log a)}\|v\|_K$.

It remains to see that the E_χ are Hölder. However, the Lyapunov decomposition is determined by the measurable section σ and the weight spaces for π with respect to A, and since $\pi(G)$ and K commute, it follows that the Lyapunov decomposition is K-invariant, and can therefore be defined by the Hölder section s instead. It follows that the E_χ are Hölder. ☐

3.2. **Proof of Theorem 3.1.** The outline of our proof follows the same basic steps as Zimmer's proof for the measurable algebraic hull in [**?**]. In the course of our proof, however, we frequently encounter situations where the gap between a measurable statement and its Hölder equivalent become significant. For example, in the proof that the unipotent radical is trivial, Zimmer can use Oseledec's Theorem to see that a certain cocycle is exponential, but there seems to be no Hölder analog for this. In these situations, we employ Theorem 2.1 as well as finite dimensional representation theory to bridge these gaps.

With the following lemma, we may assume the Hölder hull is Zariski connected.

Lemma 3.8. *Suppose G acts on a principal H bundle $R \to M$ and that the Hölder algebraic hull of the G action on R is H. Then there exists a finite cover $M' \to M$ such that the Hölder algebraic hull of the G action on $R \to M'$ is H^0, the Zariski connected component of H.*

Proof. Let $M' = \dfrac{(R \times H^0\backslash H)}{H}$. Then $M' \to M$ is a finite cover, and $R \to M'$ is a principal H^0 bundle on which G acts via bundle automorphisms. Let L be the Hölder algebraic hull of the G action on $R \to M'$. Clearly $L \subset H^0$, so it remains to show $H^0 \subset L$.

By the definition of algebraic hull, there exists a Hölder G-invariant H^0-equivariant map $\Phi : R \to L\backslash H^0$. Define

$$\tilde{\Phi} : \frac{R \times H}{H^0} \to \frac{L\backslash H^0 \times H}{H^0} \cong L\backslash H$$

so that $\tilde{\Phi}([p,h]) = [\Phi(p), h]$. Here, of course, R and $L\backslash H^0$ are right H^0 spaces, and H is a left H^0 space. The equivalence of H spaces, $\dfrac{L\backslash H^0 \times H}{H^0} \cong L\backslash H$, is obtained from the obvious bijective H-equivariant map. With right multiplication by H on itself, both $\dfrac{R \times H}{H^0}$ and $\dfrac{L\backslash H^0 \times H}{H^0}$ are right H spaces, and $\tilde{\Phi}$ is an H-equivariant map.

Let $E = \text{Functions}(H^0\backslash H, L\backslash H)$, the set of *all* functions from the finite set $H^0\backslash H$ to $L\backslash H$, and define $\Psi : R \to E$ by setting $\Psi(p)[h] = \tilde{\Phi}([ph^{-1}, h])$. If $h_0 \in H^0$ then

$$\Psi(p)[h_0h] = \tilde{\Phi}([ph^{-1}h_0^{-1}, h_0h]) = \tilde{\Phi}([ph^{-1}, h]) = \Psi(p)[h]$$

so that $\Psi(p)$ is in fact well defined. If we define the H action on E so that, for $f \in E$ and $k \in H$, $(f.k)([h]) = f([hk^{-1}])k$, then Ψ is also H-equivariant:

$$(\Psi(p).k)([h]) = \Psi(p)([hk^{-1}])k = \tilde{\Phi}([pkh^{-1}, hk^{-1}])k$$

$$= \tilde{\Phi}([pkh^{-1}, h]) = \Psi(pk)([h]).$$

The rest of the argument follows exactly as in the proof of [**?**, Theorem 9.2.6]. Namely, by ergodicity of G on M and the tameness of the H action on E, the image of Ψ is contained in a single H orbit. We can therefore view $\Psi : R \to H_\phi \backslash H$ as a Hölder G-invariant H-equivariant map, where H_ϕ is the stabilizer for some $\phi \in E$, which, as the finite intersection of algebraic subgroups, is algebraic. Since H is the Hölder algebraic hull of $R \to M$ and H_ϕ is algebraic, we must have $H_\phi = H$. However, as a stabilizer, H_ϕ leaves a finite subset of $L\backslash H$ invariant, thus $L\backslash H$ must itself be finite. Hence, $H^0 \subset L$. ☐

For the remainder of the proof we fix some $\mathbb{R}$-split Cartan $A \subset G$. Let $\mathcal{R}$ be the system of roots, and $\mathcal{W}$ the Weyl group for A.

Lemma 3.9. *There exists a normally element $a \in A$ such that $(w(\log a), \beta) \neq 0$ for every $w \in \mathcal{W}$ and for every $\beta \in \mathcal{R}$.*

Proof. By structural stability, the set of normally hyperbolic elements in G is open. Therefore, there exists a normally hyperbolic element $g \in G$ which is semisimple. Let $g = ks$ be the polar decomposition of g with polar part s. Then s is normally hyperbolic since k is contained in a compact group and commutes with s. Since some conjugate of g lies in A, it follows A contains a normally hyperbolic element. By structural stability, the set of normally hyperbolic elements in A is open, hence we can find $a \in A$ that satisfies the condition above. ☐

Henceforth, we work with a fixed element $a \in A$ satisfying Lemma 3.9. Since the volume preserving Anosov actions are ergodic, the A action is ergodic by Moore's Ergodicity Theorem [**?**]. Since the Hölder algebraic hull is Zariski connected we can write $H = L \ltimes U$ with L reductive and U unipotent. Let $P \to M$ be the G-invariant reduction of the full frame bundle to an H-bundle over M. First, we show that $Z(L)$, the center of L, is compact.

Let $N = [L, L] \ltimes U$. Then H/N is abelian, so dividing out by the maximal compact subgroup $C \subset H/N$ yields $C\backslash H/N$ which is abelian and contains no compact subgroups. Since each simple factor of G has higher rank, G has Kazhdan's property. By [**?**], the measurable hull of the G action on $(P/N)/C$ must be trivial, so by Theorem 2.1, the Hölder hull is also trivial. By the following lemma, this is a contradiction unless $C = H/N$. It follows, therefore, that $Z(L)$ must be compact.

Lemma 3.10. *Let $N \subset H$ be a normal subgroup. If the Hölder hull of the G action on $P \to M$ is H, then the Hölder hull of the G action on $P/N \to M$ is H/N.*

Proof. The Hölder hull for P/N must be contained in H/N, but if there exists a G-invariant Hölder section

$$M \to \frac{P/N \times \frac{H/N}{B/N}}{H/N} \cong \frac{P \times H/B}{H},$$

where B is a proper algebraic subgroup of H containing N, then H cannot be the Hölder hull for P. ☐

It remains, therefore, only to show that U is trivial. The first step in this direction will be to show that U must be contained in the stabilizer for a particular H action. Since the measurable algebraic hull is contained in the Hölder algebraic hull, by [**?**], there exists a measurable framing of M such that the corresponding derivative cocycle has the form $\kappa(g, m)\pi(g)$ where $\pi : G \to S$ is a homomorphism and $\kappa : G \times M \to K'$ is a cocycle taking values in a compact subgroup of H commuting with $\pi(G)$.

Note that L contains the image of the measurable superrigidity homomorphism. Since the action is Anosov and π determines the Lyapunov splitting, the image $\pi(G)$ is a noncompact semisimple Lie group. Hence, L is not compact.

Let $L = ZS$ where S is semisimple with no compact factors, Z is compact and centralizes S, and the product is almost direct. If $\mathfrak{U}$ is the Lie algebra of U, then we have a representation $\rho : L \to GL(\mathfrak{U})$ obtained from the semidirect product. We will denote the restriction of ρ to S simply as ρ. Also let $\sigma : S \to GL(n, \mathbb{R})$ be the representation determined by the embedding $S \subset H \subset SL(n, \mathbb{R})$.

Lemma 3.11. *If $U \neq 0$, then $\rho|_S$ is not trivial.*

Proof. If $\rho|_S$ is trivial, then we obtain an amenable algebraic group $H/(\ker(\rho) \ltimes [U, U]) \cong Z_1 \ltimes U/[U, U]$, where Z_1 is some compact quotient of Z. By Kazhdan's property, the measurable hull of $P/(\ker(\rho) \ltimes [U, U])$ must be contained in the compact part of this amenable group [**?**]. If we could show that the Hölder hull must also be contained in the compact part, Lemma 3.10 would then force $U/[U, U] = 0$, so that $U = 0$. As Z_1 is compact, there is a $Z_1 \ltimes U/[U, U]$-invariant metric on $V = (Z_1 \ltimes U/[U, U])/Z_1$. Hence Theorem 2.1 applied to this V establishes this, thereby completing the proof. ☐

Let $\{w_1, \ldots, w_t\}$ be some fixed ordering of all the elements in $\mathcal{W}$, and let k_i be the dimension of $E^-_{w_i(a)}(m)$ and let l_i be the dimension of $E^+_{w_i(a)}(m)$. Let

$$V = Gr(k_1, n) \times Gr(l_1, n) \times Gr(k_2, n) \times Gr(l_2, n) \times \cdots \times Gr(l_t, n),$$

and define a Hölder A-invariant section $\omega : M \to E_V$ so that

$$\omega(m) = (E^-_{w_1(a)}(m), E^+_{w_1(a)}(m), \ldots, E^-_{w_t(a)}(m), E^+_{w_t(a)}(m)).$$

Then ω corresponds to a Hölder A-invariant H-equivariant map $\Omega : P \to V$. Dividing out by the H-action, we obtain an A-invariant map $\tilde{\Omega} : P/H \cong M \to V/H$. Note that H acts tamely on V [?]. By ergodicity of A on M, $\tilde{\Omega}$ is constant, and therefore Ω is contained in an H-orbit. Thus, we may consider Ω as a map from P into $H_{x_0}\backslash H$ where H_{x_0} is the stabilizer in H of a point $x_0 \in V$.

We want to use Ω to produce a G-invariant H-equivariant map, which, in turn, will allow us to calculate the algebraic hull. Let Z be the center of G and note that Ω is Z-invariant. Thus, we can define $\Omega' : P \times (G/ZA) \to H_{x_0}\backslash H$ so that $\Omega'(p, g) = \Omega(g^{-1}p)$. Letting G act on $P \times (G/ZA)$ via the diagonal action, Ω' becomes G-invariant. Let $F(G/ZA, H_{x_0}\backslash H)$ be the space of measurable functions with the topology of convergence in measure, and define

$$\Psi : P \to F(G/ZA, H_{x_0}\backslash H)$$

by $\Psi(p)(g) = \Omega'(p, g) = \Omega(g^{-1}p)$.

Following the proof of Step 2 in the proof of measurable superrigidity in [?] and the proof of Lemma 3.3 in [?], we see that $\Psi(p)$ is a rational function for almost every p, i.e., $\Psi : P \to R = \mathrm{Rat}(G/ZA, H_{x_0}\backslash H)$. Let G and H act on R so that

$$(g_1.r)(g) = r(g_1^{-1}g), \text{ and}$$

$$(r.h)(g) = r(g).h.$$

Then Ψ is G and H equivariant. By H-equivariance of Ψ, the degree of $\Psi(p)$ as a rational function depends only on the base point of p in M. Since G acts ergodically on M, it follows that these rational functions must have the same degree almost everywhere. But, the set of rational functions of a fixed degree is closed, so by continuity we may conclude that $\Psi(p)$ is rational for every p.

Again, following the proof of Step 3 for measurable superrigidity in [?] and the comments in [?] we can conclude that Ψ is G-invariant and contained in a single H orbit in R, i.e., $\Psi : P \to H_r\backslash H$ where H_r is the stabilizer in H for some $r \in R$. Since H_r consists of the subgroup of H pointwise fixing the image of r, and since without loss of generality we may assume that $x_0 \in V$ lies in the image of r, we have that $H_r \subset H_{x_0}$. However, by the definition of algebraic hull we must have $U \subset H_r$, and consequently $U \subset H_{x_0}$.

To complete the proof of Theorem 3.1, we will use $U \subset H_{x_0}$ to show that U must be trivial. Since $U \subset H \subset SL(n, \mathbb{R})$, we have a natural action of $\mathfrak{U}$ on $\mathbb{R}^n$. Set

$$V_0 = \{v \in \mathbb{R}^n | Xv = 0 \text{ for all } X \in \mathfrak{U}\}.$$

Since L normalizes U, L leaves V_0 invariant. Since L is reductive with compact center there is a splitting $\mathbb{R}^n = V_0 \bigoplus V_1$ into L-invariant subspaces. Since any subgroup of $GL(n)$ fixing all of $\mathbb{R}^n$ must be trivial, if we can show that $V_0 = \mathbb{R}^n$, then U must be trivial. Hence, we must show that $V_1 = 0$. To do this, we will show that all of $\mathfrak{U}$ kills the maximal weight space for any nontrivial irreducible subrepresentation of $\sigma \circ \pi$ on V_1. This implies that the given nontrivial irreducible subrepresentation cannot lie in V_1, and that $\sigma \circ \pi|_{V_1}$ is a sum of trivial representations. By the next lemma, the proof of Theorem 3.1 is complete once we see that $\sigma \circ \pi|_{V_1}$ is a sum of trivial representations.

Lemma 3.12. *If $\sigma \circ \pi|_{V_1}$ is a sum of trivial representations, then $V_1 = 0$.*

Proof. For $v \in V_1 \setminus \{0\}$, the framing determines a measurable vector field $\mathcal{V}$ on M. Since S fixes v, we have $dg_m(\mathcal{V}(m)) = \kappa(g, m)\mathcal{V}(gm)$. In particular, with $g = a^n$ and using the recurrence properties of a, it follows that $\mathcal{V}$ has 0 Lyapunov exponent. Consequently, $\mathcal{V}$ must lie in the M_0A tangent direction, where M_0 is the compact part of the centralizer of A in G (cf. the beginning of Section 2.1). Note that K' leaves the 0 Lyapunov direction invariant and therefore the M_0A tangent direction invariant.

Fix a basis consisting of left invariant vector fields on $\mathfrak{g}$. They determine a framing of the tangent directions to the G orbit on M which under G transforms according to the adjoint representation (cf. the beginning of Section 2). Since the adjoint orbit of any nonzero element in $\mathfrak{m}_0 \oplus \mathfrak{a}$ cannot be contained in $\mathfrak{m}_0 \oplus \mathfrak{a}$, there exists some

$g \in G$ such that $dg(\mathcal{V}(m))$ cannot lie in the $M_0 A$ direction. This contradiction establishes that there cannot be nonzero elements in V_1. □

To show that $\sigma \circ \pi|_{V_1}$ is a sum of trivial representations, we shall need the following three results.

Lemma 3.13. *Suppose $X \in \mathfrak{U}$ has $\rho \circ \pi$ weight ν, and $v \in V_1$ has $\sigma \circ \pi$ weight χ. Then either $X.v = 0$ or v has $\sigma \circ \pi$ weight $\nu + \chi$.*

Proof. Let b be in $\mathfrak{a}$, the Lie algebra of A. Then we have $\pi(b)(X.v) = (X\pi(b) + [\pi(b), X]).v = X(\pi(b).v) + [\pi(b), X].v = X(\chi \circ \pi(b).v) + (\nu \circ \pi(b))X.v = (\chi \circ \pi(b) + \nu \circ \pi(b))X.v$. □

It is important to note that $X.v$ may well lie in a different irreducible subrepresentation than v.

Lemma 3.14. *Let $X \in \mathfrak{U}$, let K_a be the hyperplane $\{\alpha \in \mathfrak{a}^* | \alpha(\log a) = 0\}$, let $K_a^+ = \{\beta \in \mathfrak{a}^* | (\alpha, \log a) > 0\}$, and $K_a^- = \{\beta \in \mathfrak{a}^* | (\alpha, \log a) < 0\}$. If $v \in \mathbb{R}^n$ is a weight vector with weight in $K_{w(a)}^{\pm}$ for any $w \in \mathcal{W}$, then either $Xv \in K_{w(a)}^{\pm}$ or $Xv = 0$.*

Proof. Note that $x_0 \in V$ corresponds to a $2t$-tuple of linear subspaces

$$(E_{w_1(a)}^-, E_{w_1(a)}^+, \ldots, E_{w_t(a)}^-, E_{w_t(a)}^+).$$

Since $U \subset H_{x_0}$, $X \in \mathfrak{U}$ implies that X preserves x_0, and therefore the stable and unstable directions for all $w_i(a)$ as well. □

Lemma 3.15. *Let N be a root for $\mathfrak{S}$, the Lie algebra of S, and suppose $[X, N] = 0$ for some $X \in \mathfrak{U}$. If $Xv = 0$, then $X(Nv) = 0$*

Proof. This follows since $X(Nv) = [X, N]v + N(Xv) = 0.v + N(0) = 0$. □

Consider an irreducible subrepresentation of $\rho : S \to GL(\mathfrak{U})$ with maximal weight μ. Then for any weight vector X in this irreducible, its weight has the form $\mu - \sum_i l_i \alpha_i$ where the α_i are the positive simple roots and the l_i are nonnegative integers. This is well known in the complex case [?]. For the real case, simply consider the complexification. The only problem is caused by imaginary root vectors R_β which show up in the real case. However, we may commute any imaginary R_β past all nonimaginary root vectors R_α since $[R_\beta, R_\alpha]$ is nonimaginary. Since the maximal weight space W_μ of A is invariant under the imaginary root vectors, the claim follows in the real case. Moreover X can be written in the form

$$X = [R_{-\alpha_{i_n}}[\cdots[R_{-\alpha_{i_1}}, X_\mu]\cdots],$$

for some weight vector X_μ with weight μ and the R_{α_i} nonimaginary root vectors. Expanding out the brackets, we obtain

$$X = \sum_i N_i X_\mu M_i,$$

where N_i and M_i have the form $R_{-\alpha_{i_1}} \cdots R_{-\alpha_{i_n}}$ for varying choices of $\{\alpha_i\}$. So to see that all of $\mathfrak{U}$ kills the maximal weight space for any nontrivial subrepresentation of σ on V_1, it clearly suffices to show that X_μ kills all the weight spaces in such an irreducible subrepresentation of σ on V_1.

Let μ be the maximal weight for an irreducible subrepresentation of ρ, and let α be the maximal root for S. In particular, we have that $\langle \mu, \alpha \rangle > 0$. Let K_a, K_a^+, K_a^- be as in Lemma 3.14. By renaming if necessary, we may assume that the normal vector to K_a lies in the positive Weyl chamber.

Lemma 3.16. *Suppose β is a weight for σ such that $\beta \in K_a^+$ and $w_\alpha(\beta) \in K_a^-$. Let $w_\alpha \in \mathcal{W}$ be defined as the reflection through the hyperplane perpendicular to α. Then there exists a weight λ in the α string from $w_\alpha(\beta)$ to β such that either*

1. *$\lambda \in K_a^-$ and $\lambda + \mu \in K_a^+$, or,*
2. *$\lambda + \alpha \in K_a^+$ and $\lambda + \alpha + w_\alpha(\mu) \in K_a^-$.*

Further, this result holds if we replace K_a with $K_{w(a)}$ for any $w \in \mathcal{W}$.

Proof. Since $\beta \in K_a^+$ and $w_\alpha(\beta) \in K_a^-$, the α-string from β to $w_\alpha(\beta)$ intersects K_a, say between the weights λ and $\lambda+\alpha$. Form the triangle with vertices $\lambda, \lambda+\mu$, and $\lambda+\mu-w_\alpha(\mu)$. Since $\mu - w_\alpha(\mu) = \langle \alpha, \mu \rangle \alpha$ is a positive integral multiple of α, it follows that $\lambda+\alpha$ lies on the edge of this triangle with vertices λ and $\lambda+\mu-w_\alpha(\mu)$. Since K_a is a hyperplane which intersects this side of the triangle, it must also intersect one of the other sides, i.e., either K_a intersects the line between λ and $\lambda+\mu$, or intersects the line between $\lambda+\mu$ and $\lambda+\mu-w_\alpha(\mu)$. The former case satisfies (1). If the latter case holds, then as K_a intersects the line between λ and $\lambda+\alpha$ and the

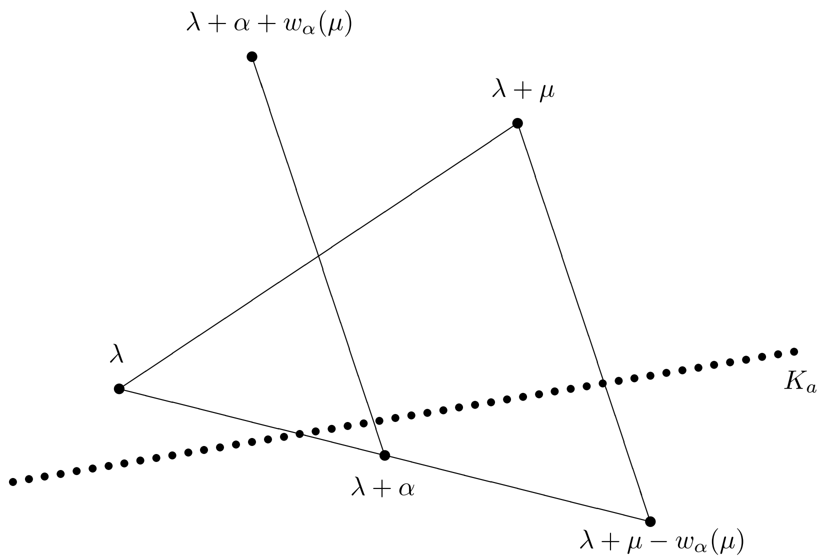

FIGURE 1. The hyperplane K_a and the triangle with vertices $\lambda, \lambda+\mu$, and $\lambda+\mu-w_\alpha(\mu)$.

line between $\lambda+\mu$ and $\lambda+\mu-w_\alpha(\mu)$, K_a must also intersect the line between $\lambda+\alpha$ and $\lambda+\alpha+w_\alpha(\mu)$ (see Figure 1). This satisfies (2). ☐

Lemma 3.17. *Suppose β is a weight for σ such that $\beta \in K_a^+$ and $w_\alpha(\beta) \in K_a^-$. Then for every $v \in V_\beta \subset \mathbb{R}^n$ and for every maximal weight vector X in our given irreducible subrepresentation of ρ, $Xv = 0$. Further, this result holds if we replace K_a with $K_{w(a)}$ for any $w \in \mathcal{W}$.*

Proof. Let λ be as in Lemma 3.16, and assume that the first of the two possible conditions holds. Since $\lambda \in K_a^-$ and $\lambda+\mu \in K_a^+$, Lemma 3.14 implies that $Xw = 0$ for any weight vector $w \in \mathbb{R}^n$ with weight λ and for any weight vector $X \in \mathfrak{U}$ with weight μ. For $v \in V_\beta$, there exists a series of root vectors $\{R_i\}$ with root α and $w \in V_\lambda$ such that $(\sigma\pi(R_l)) \circ \cdots \circ (\sigma\pi(R_1))(w) = v$. Since $\rho\pi(R_i)(X) = [R_i, X] = 0$, we have

$$Xv = X(\sigma\pi(R_l)) \circ \cdots \circ (\sigma\pi(R_1))(w) = (\sigma\pi(R_l)) \circ \cdots \circ (\sigma\pi(R_1))(Xw) = 0.$$

Suppose the latter case of Lemma 3.16 holds. Choose $g_\alpha \in G$ such that $Ad(g_\alpha)$ corresponds to w_α. An argument similar to the one above shows that for any $v \in V_{w_\alpha(\beta)}$ and any maximal ρ weight vector X, $(Ad(g_\alpha)(X))v = 0$. Note that $\sigma\pi(g_\alpha^{-1})(V_\beta) \subset V_{w_\alpha(\beta)}$. Hence, for any $w \in V_\beta$, we can write $w = \sigma\pi(g_\alpha)(v)$ for some $v \in V_{w_\alpha(\beta)}$. Finally, as ρ comes from the adjoint representation of H, we have $\sigma(\rho(g_\alpha)(X)) = \sigma(g_\alpha)(\sigma(X))\sigma(g_\alpha^{-1})$. Thus,

$$0 = (Ad(g_\alpha)(X))v = \sigma(\rho(g_\alpha(X))v = \sigma(g_\alpha)\sigma(X)\sigma(g_\alpha^{-1})\sigma(g_\alpha)v = \sigma(g_\alpha)\sigma(X)v = \sigma(g_\alpha)(Xv).$$

Since $\sigma(g_\alpha)$ is invertible, we have that $Xv = 0$. ☐

The previous lemma shows that certain types of weight spaces are killed by all weight vectors $X \in \mathfrak{U}$ with weight μ. In the remainder of the proof, we first show that the maximal weight space V_χ for any irreducible subrepresentation of σ is killed by all such X's, and second, that this is sufficient to see that all weight spaces for σ are killed by any such X.

Proposition 3.18. *Let $\mathcal{R}$ be a root system with $\mathcal{C}^+$ the positive Weyl chamber, α the maximal root, $\mathcal{W}$ the Weyl group, and w_α the element of $\mathcal{W}$ defined as reflection through the hyperplane perpendicular to α. If $\chi \in \overline{\mathcal{C}^+}$ is a weight, then there exists $w \in \mathcal{W}$ such that $(\chi, w(a))$ and $(w_\alpha(\chi), w(a))$ have opposite signs.*

Proof. Let $\mathcal{P}$ be the 2 dimensional space spanned by χ and α (see Figure 2). Then $\chi \in \mathcal{C}^+ \cap \mathcal{P}$ and $w_\alpha(\chi) \in$

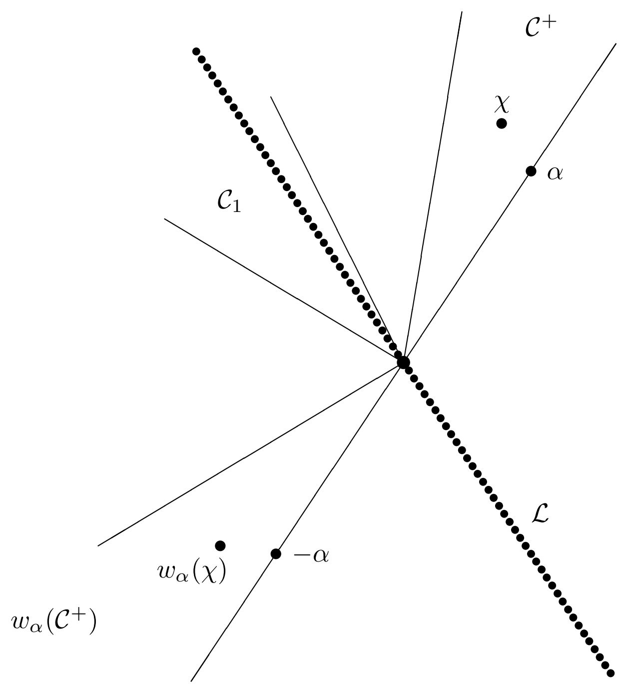

FIGURE 2. The plane $\mathcal{P}$ spanned by χ and α.

$w_\alpha(\mathcal{C}^+) \cap \mathcal{P}$. Moreover, since $(\nu, \alpha) > 0$ for every $\nu \in \overline{\mathcal{C}^+}$, $\overline{\mathcal{C}^+} \cap \mathcal{P}$ and $w_\alpha(\overline{\mathcal{C}^+}) \cap \mathcal{P}$ cannot share an edge. Hence, there exists a half cone in $\mathcal{P}$ which separates $\mathcal{C}^+ \cap \mathcal{P}$ and $w_\alpha(\mathcal{C}^+) \cap \mathcal{P}$ and which intersects the line segment joining χ and $w_\alpha(\chi)$. Note that such a half cone consists of the intersections of Weyl chambers with $\mathcal{P}$. Hence we can find some Weyl chamber $\mathcal{C}_1$ such that $\mathcal{C}_1$ intersects the line segment joining χ and $w_\alpha(\chi)$.

If a is normally hyperbolic, then so is $\omega(a)$ for every $\omega \in \mathcal{W}$. Thus, using the transitivity of $\mathcal{W}$ on the set of Weyl chambers, we can find some $w \in \mathcal{W}$ such that $K_{w(a)}$ intersects $\mathcal{C}_1$ which, since $K_{w(a)}$ has codimension 1, must intersect $\mathcal{P}$ in at least a line. Let $\mathcal{L} = K_{w(a)} \cap \mathcal{P}$. If $\mathcal{L} = \mathcal{P}$, then $w(a)$ is perpendicular to α. This cannot happen since we have chosen a such that $(\omega(a), \beta) \neq 0$ for every $\omega \in \mathcal{W}$ and for every root β.

We may therefore assume that $\mathcal{L}$ is a line. Since $\mathcal{C}_1$ intersects the line segment between χ and $w_\alpha(\chi)$, it follows that $\mathcal{L}$ does as well. Hence $(\chi, w(a))$ and $(w_\alpha(\chi), w(a))$ have opposite signs. This is possible since $w(a)$ is conjugate to a in G and hence normally hyperbolic. ☐

Corollary 3.19. *If V_χ is a maximal weight space for any irreducible subrepresentation of σ, then $Xv = 0$ for every $v \in V_\chi$ and for every weight vector $X \in \mathfrak{U}$ with weight μ.*

Proof. Proposition 3.18 shows that, by possibly having to replace a with $-a$, there exists some $w \in \mathcal{W}$ such that $\chi \in K^+_{w(a)}$ and $w_\alpha(\chi) \in K^-_{w(a)}$. Now apply Lemma 3.17 using $K^\pm_{w(a)}$ in place of $K^\pm_a$. ☐

Let $V_\nu \subset \mathfrak{U}$ be any weight space in σ with weight ν. We will complete the proof by showing that any $X \in V_\mu$ kills any $v \in V_\nu$. Assume for the sake of contradiction that $Xv \neq 0$. Let $\{\chi_i\}$ be the set of the maximal weights for all the irreducible subrepresentations for σ. From Proposition 3.18, we can choose $a_i \in \mathcal{W}(\pm a)$ to be a normally hyperbolic element in G such that $\chi_i \in K^+_{a_i} \cap K^-_{w_\alpha(a_i)}$. If $\nu \in K^+_{a_i} \cap K^-_{w_\alpha(a_i)}$ for any i, then Lemma 3.17 would imply $Xv = 0$. Hence we can assume that $\nu \notin \mathcal{K} = \bigcup_i (K^+_{a_i} \cap K^-_{w_\alpha(a_i)})$.

Further, if $\nu + \mu \in \mathcal{K}$, then by Lemma 3.14 we would have $Xv = 0$. Thus $\nu + \mu \notin \mathcal{K}$, and we have that Xv cannot be a maximal weight vector for any irreducible subrepresentation of σ. Hence it is possible to find some positive simple root vector R_1 such that $\sigma\pi(R_1)(Xv) \neq 0$. Let $v_1 = \sigma\pi(R_1)v$. Then v_1 is a weight vector with some weight ν_1, such that with respect to the usual ordering of weights, $\nu \prec \nu_1$. Since $\rho\pi(R_1)(X) = 0$, we have $Xv_1 = X(\rho\pi(R_1)(v)) = \sigma\pi(R_1)(Xv) \neq 0$. We repeat this process using v_1 instead of v. Since $Xv_1 \neq 0$, it follows that both ν_1 and $\nu_1 + \mu$ do not lie in $\mathcal{K}$. Therefore, Xv_1 cannot be a maximal weight for any irreducible subrepresentation of σ. Hence we can find a positive simple root R_2 such that $\sigma\pi(R_2)(Xv_1) \neq 0$. If $v_2 = \sigma\pi(R_2)(v_1)$, then it follows that $Xv_2 \neq 0$, and v_2 is a weight vector with some weight ν_2 such that $\nu \prec \nu_1 \prec \nu_2$. We can continue this process and produce an infinite sequence of weight vectors each with a different weight. Obviously, this contradicts the finite dimensionality of σ. Hence our assumption that $Xv \neq 0$ must be false. Theorem 3.1 now follows. □

3.3. **Proof of Proposition 3.3.** As mentioned before the statement of Proposition 3.3, by setting $\phi(m) = E_a^-(m)$ we have an A-invariant Hölder section of $E_{Gr(k,n)} \to M$. Let $\Phi : P \to Gr(k,n)$ be the H-equivariant map corresponding to ϕ.

Lemma 3.20. *The section ϕ is invariant under some parabolic subgroup of G.*

Proof. If $U^- = \{u \in G \mid Ad(a^n)u \to 1 \text{ as } n \to \infty\}$, then $Q = Z_G(A) \cdot U^-$ is a parabolic subgroup. To see that ϕ is Q-invariant note that for a fixed $q \in Q$, $\{a^n q a^{-n}\}$ is bounded in G. Hence, for $v \in E_a^-(m)$, $d(a^n)d(q)v = d(a^n q a^{-n})d(a^n)v \to 0$ as $n \to \infty$ showing that ϕ is indeed Q-invariant. □

Proposition 3.3 now follows by letting K be the kernel of the H action on $\Phi(P)$. □

3.4. **Proof of Theorem 3.6.** The Theorem follows immediately by combining Proposition 3.3 and Theorem 3.4. It remains only to see that K is compact when the irreducible subrepresentations of π are multiplicity free.

Let H be the Hölder hull, so by Theorem 3.1, H is reductive with compact center. Write $H = CS$ as an almost direct product where C is compact and S is semisimple with no compact factors. Since G has no compact factors, $\pi(G)$ must take values in S. Since K is the kernel of the H action on $\Phi(P)$, it commutes with the image of $\pi(G)$. Using Schur's Lemma, and the assumption of multiplicity free, $K \cap S$ must be abelian. Thus, $K \cap S$ has a trivial connected component, i.e., we have that $K \subset C$. □

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